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ON FLECK QUOTIENTS

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ABSTRACT. Let p be a prime, and let $n \geq 1$ and r be integers. In this paper we study Fleck's quotient

$$F_p(n, r) = (-p)^{-\lfloor (n-1)/(p-1) \rfloor} \sum_{k \equiv r \pmod{p}} \binom{n}{k} (-1)^k \in \mathbb{Z}.$$

We determine $F_p(n, r) \pmod{p}$ completely by certain number-theoretic and combinatorial methods; consequently, if $2 \leq n \leq p$ then

$$\sum_{k=1}^n (-1)^{pk-1} \binom{pn-1}{pk-1} \equiv (n-1)! B_{p-n} p^n \pmod{p^{n+1}},$$

where B_0, B_1, \dots are Bernoulli numbers. We also establish the Kummer-type congruence $F_p(n+p^a(p-1), r) \equiv F_p(n, r) \pmod{p^a}$ for $a = 1, 2, 3, \dots$, and reveal some connections between Fleck's quotients and class numbers of the quadratic fields $\mathbb{Q}(\sqrt{\pm p})$ and the p -th cyclotomic field $\mathbb{Q}(\zeta_p)$. In addition, generalized Fleck quotients are also studied in this paper.

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1. INTRODUCTION AND MAIN RESULTS

Let $m \in \mathbb{Z}^+ = \{1, 2, \dots\}$, $n \in \mathbb{N} = \{0, 1, \dots\}$ and $r \in \mathbb{Z}$, and define

$$C_m(n, r) = \sum_{k \equiv r \pmod{m}} \binom{n}{k} (-1)^k. \quad (1.0)$$

This sum has been studied by various authors and many applications have been found (cf. [S02] and its references). The following well-known observation is fundamental:

$$mC_m(n, r) = \sum_{k=0}^n \binom{n}{k} (-1)^k \sum_{\gamma^m=1} \gamma^{k-r} = \sum_{\gamma^m=1} \gamma^{-r} (1 - \gamma)^n.$$

Note that

$$C_m(n+1, r) = C_m(n, r) - C_m(n, r-1)$$

since $x^{-r}(1-x)^{n+1} = x^{-r}(1-x)^n - x^{-r+1}(1-x)^n$.

Let p be a prime, and let $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. In 1913 A. Fleck (cf. [D, p. 274]) showed that

$$\text{ord}_p(C_p(n, r)) \geq \left\lfloor \frac{n-1}{p-1} \right\rfloor,$$

where $\text{ord}_p(\alpha)$ denotes the p -adic order of a p -adic number α , and $\lfloor \cdot \rfloor$ is the well-known floor function. Fleck's result is fundamental in the recent investigation of the ψ -operator related to Fontaine's theory, Iwasawa's theory, and p -adic Langlands correspondence (cf. [Co], [SW] and [W]); it also plays an indispensable role in Davis and Sun's study of homotopy exponents of special unitary groups (cf. [DS] and [SD]). In this paper we are interested in the *Fleck quotient*

$$F_p(n, r) := (-p)^{-\lfloor (n-1)/(p-1) \rfloor} C_p(n, r) + \llbracket n = 0 \rrbracket. \quad (1.1)$$

(Throughout this paper, for an assertion A we let $\llbracket A \rrbracket$ take 1 or 0 according as A holds or not.)

For $a \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, we use $\{a\}_m$ to denote the least nonnegative residue of $a \bmod m$ (thus $\{a\}_m/m$ is the fractional part $\{a/m\}$ of a/m). For a prime p and an integer a , we define $q_p(a) = (a^{p-1} - 1)/p$ which is an integer if $a \not\equiv 0 \pmod{p}$.

By a number-theoretic approach related to Gauss sums, we establish the following explicit result.

Theorem 1.1. *Let p be a prime, and let $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Set $n_0 = \{n\}_p$ and $n_1 = \{n_0 - n\}_{p-1} = \{-\lfloor n/p \rfloor\}_{p-1}$. If $n_0 \leq n_1$, then*

$$F_p(n, r) \equiv \frac{(-1)^{n_1}}{n_1!} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k - r)^{n_1} \pmod{p}. \quad (1.2)$$

If $n_0 > n_1 = 0$, then

$$F_p(n, r) \equiv (-1)^{\{r\}_p} \binom{n_0}{\{r\}_p} \pmod{p}. \quad (1.3)$$

If $n_0 > n_1 > 0$, then

$$F_p(n, r) \equiv \frac{(-1)^{n_1-1}}{(n_1-1)!} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{n_1} q_p(k-r) \pmod{p}. \quad (1.4)$$

Corollary 1.1. *Let p be a prime and let $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then*

$$F_p(pn, r) \equiv \frac{r^{n^*}}{n^*!} \pmod{p} \quad (1.5)$$

where $n^* = \{-n\}_{p-1}$. Consequently,

$$F_p\left(p\frac{p-1}{2}, r\right) \equiv \begin{cases} (-1)^{(h(-p)+1)/2} \left(\frac{r}{p}\right) \pmod{p} & \text{if } p \neq 3 \text{ \& } 4 \mid p+1, \\ (-1)^{(h(p)-1)/2} \left(\frac{r}{p}\right) \frac{v}{2} \pmod{p} & \text{if } 4 \mid p-1, \end{cases} \quad (1.6)$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol, and $h(-p)$ and $h(p)$ are the class numbers of the quadratic fields $\mathbb{Q}(\sqrt{-p})$ and $\mathbb{Q}(\sqrt{p})$ respectively, and for $p \equiv 1 \pmod{4}$ we write the fundamental unit of $\mathbb{Q}(\sqrt{p})$ in the form $(v+u\sqrt{p})/2$ with $u, v \in \mathbb{Z}$ and $u \equiv v \pmod{2}$.

Proof. Note that $\{pn\}_p = 0$. By Theorem 1.1,

$$F_p(pn, r) \equiv \frac{(-1)^{n^*}}{n^*!} \sum_{k=0}^0 \binom{0}{k} (-1)^k (k-r)^{n^*} = \frac{r^{n^*}}{n^*!} \pmod{p}.$$

When $p \neq 2$ and $n = (p-1)/2$, we have $n^* = (p-1)/2$ and hence

$$\begin{aligned} F_p\left(p\frac{p-1}{2}, r\right) &\equiv r^{(p-1)/2} (-1)^{(p-1)/2} \frac{((p-1)/2)!}{\prod_{k=1}^{(p-1)/2} k(p-k)} \\ &\equiv \left(\frac{r}{p}\right) (-1)^{(p-1)/2} \frac{((p-1)/2)!}{(p-1)!} \quad (\text{by Euler's criterion}) \\ &\equiv (-1)^{(p+1)/2} \left(\frac{r}{p}\right) \frac{p-1}{2}! \pmod{p} \quad (\text{by Wilson's theorem}). \end{aligned}$$

If $p > 3$ and $p \equiv 3 \pmod{4}$, then

$$\frac{p-1}{2}! \equiv (-1)^{(h(-p)+1)/2} \pmod{p}$$

by a result of L. J. Mordell [M]. When $p \equiv 1 \pmod{4}$ and $\varepsilon_p = (v + u\sqrt{p})/2 > 1$ is the fundamental unit of $\mathbb{Q}(\sqrt{p})$ with $u, v \in \mathbb{Z}$ and $u \equiv v \pmod{2}$, by S. Chowla [C] we have

$$\frac{p-1}{2}! \equiv (-1)^{(h(p)+1)/2} \frac{v}{2} \pmod{p}.$$

Combining the above we immediately obtain (1.6). \square

Remark. Let n be a positive integer and $p > 2n + 1$ be a prime. By the first part of Corollary 1.1 in the case $r = 0$, we have

$$\binom{2pn}{pn}(-1)^n + 2 \sum_{k=0}^{n-1} \binom{2pn}{pk}(-1)^k = \sum_{k=0}^{2n} \binom{2pn}{pk}(-1)^{pk} \equiv 0 \pmod{p^{2n+1}}$$

and hence

$$\binom{2pn-1}{pn-1} = \frac{1}{2} \binom{2pn}{pn} \equiv \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{2pn}{pk} \pmod{p^{2n+1}}. \quad (1.7)$$

When $n = 1$ and $p > 3$, this gives the Wolstenholme congruence

$$\frac{1}{2} \binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

When $n = 2$ and $p > 5$, (1.7) yields the following new congruence

$$\binom{4p-1}{2p-1} = \frac{1}{2} \binom{4p}{2p} \equiv \binom{4p}{p} - 1 \pmod{p^5}.$$

Our second approach to Fleck quotients is of combinatorial nature. It involves Stirling numbers of the second kind as well as higher-order Bernoulli polynomials.

Let $n \in \mathbb{N}$. The Stirling numbers $S(n, k)$ ($k \in \mathbb{N}$) of the second kind are given by

$$x^n = \sum_{k \in \mathbb{N}} S(n, k)(x)_k,$$

where

$$(x)_0 = 1 \quad \text{and} \quad (x)_k = x(x-1) \cdots (x-k+1) \quad \text{for } k = 1, 2, \dots$$

Clearly, $S(n, n) = 1$, and $S(n, k) = 0$ if $k > n$. When $n + k > 0$, $S(n, k)$ is actually the number of ways to partition a set of cardinality n into k

nonempty subsets. Here is an explicit formula (cf. [LW, p. 126]) for Stirling numbers of the second kind:

$$S(n, k) = \frac{1}{k!} \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} j^n.$$

As $S(i, k) = 0$ for all those $i \in \mathbb{N}$ with $i < k$, we have *Euler's identity*

$$\sum_{j=0}^k \binom{k}{j} (-1)^j P(j) = 0,$$

where $P(x)$ is any polynomial with $\deg P < k$ having complex number coefficients. It is known (cf. [LW, p. 126]) that

$$\sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!};$$

in other words,

$$(e^x - 1)^k = \sum_{n=k}^{\infty} \bar{S}(n, k) x^n \quad \text{with} \quad \bar{S}(n, k) = \frac{k!}{n!} S(n, k).$$

For $m = 0, 1, \dots$, the m -th order Bernoulli polynomials $B_n^{(m)}(t)$ ($n \in \mathbb{N}$) are defined by

$$\frac{x^m e^{tx}}{(e^x - 1)^m} = \sum_{n=0}^{\infty} B_n^{(m)}(t) \frac{x^n}{n!}, \quad (1.8)$$

and those $B_n^{(m)} = B_n^{(m)}(0)$ are called the m -th order Bernoulli numbers. The usual Bernoulli polynomials and numbers are $B_n(t) = B_n^{(1)}(t)$ and $B_n = B_n(0) = B_n^{(1)}$ respectively. (It is well known that $B_0 = 1$, $B_1 = -1/2$ and $B_{2k+1} = 0$ for $k = 1, 2, \dots$; the reader may consult [IR, pp. 228–248] for the basic properties of Bernoulli numbers.) For a formal power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we use $[x^n]f(x)$ to denote the coefficient a_n of the monomial x^n in $f(x)$. Thus

$$\begin{aligned} B_n^{(m)}(t) &= [x^n] n! \left(\frac{x}{e^x - 1} \right)^m e^{tx} \\ &= [x^n] n! \sum_{k=0}^{\infty} B_k^{(m)} \frac{x^k}{k!} \sum_{j=0}^{\infty} \frac{(tx)^j}{j!} = \sum_{k=0}^n \binom{n}{k} B_k^{(m)} t^{n-k}. \end{aligned}$$

It is also easy to verify that $B_n^{(m)}(m-t) = (-1)^n B_n^{(m)}(t)$, and

$$\frac{B_n^{(m)}(t)}{n!} = \sum_{k_0 + \dots + k_{m-1} = n} \frac{B_{k_0}(t)}{k_0!} \prod_{0 < i < m} \frac{B_{k_i}}{k_i!} \quad \text{provided } m > 0.$$

If $0 \leq n < p-1$, then B_0, \dots, B_n are p -adic integers by the von Staudt-Clausen theorem (cf. [IR, p. 233]) or the recurrence $\sum_{k=0}^l \binom{l+1}{k} B_k = 0$ ($l = 1, 2, \dots$), therefore $B_n^{(m)}(t) \in \mathbb{Z}_p[t]$ where \mathbb{Z}_p is the ring of p -adic integers.

Our discovery of the next theorem was actually motivated by Theorem 1.1.

Theorem 1.2. *Let p be a prime, and let $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Set $n^* = \{-n\}_{p-1}$. For any integer $m \equiv n \pmod{p}$, if $m \geq 0$ then $(-1)^n F_p(n, r)$ is congruent to*

$$\begin{aligned} \sum_{k=0}^{n^*} \bar{S}(n^* - k + m, m) \frac{(-r)^k}{k!} &= \sum_{k=0}^{n^*} \bar{S}(m + n^*, m + k) \binom{-r}{k} \\ &= \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \frac{(k-r)^{m+n^*}}{(m+n^*)!} \end{aligned} \quad (1.9)$$

modulo p ; if $m \leq 0$ then we have

$$F_p(n, r) \equiv \frac{(-1)^{n^*}}{n^*!} B_{n^*}^{(-m)}(-r) \equiv -(p-1-n^*)! B_{n^*}^{(-m)}(-r) \pmod{p}. \quad (1.10)$$

The following consequence determines $B_n^{(m)}(a)$ modulo a prime p for $m \in \{1, \dots, p\}$, $n \in \{0, \dots, p-2\}$ and $a \in \mathbb{Z}$.

Corollary 1.2. *Let p be a prime and $r \in \mathbb{Z}$. Let $n_0 \in \{0, \dots, p-1\}$ and $n_1 \in \{0, \dots, p-2\}$. If $n_0 \leq n_1$, then*

$$B_{n_1-n_0}^{(p-n_0)}(-r) \equiv \frac{1}{(n_1)_{n_0}} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^{n_0-k} (k-r)^{n_1} \pmod{p}. \quad (1.11)$$

If $n_0 > n_1 = 0$, then

$$B_{p-n_0+n_1-1}^{(p-n_0)}(-r) \equiv \frac{(-1)^{\{r\}_p-1}}{n_0!} \binom{n_0}{\{r\}_p} \pmod{p}. \quad (1.12)$$

If $n_0 > n_1 > 0$, then

$$\begin{aligned} B_{p-n_0+n_1-1}^{(p-n_0)}(-r) &\equiv \frac{(-1)^{n_1}}{(n_0-n_1)!(n_1-1)!} \\ &\quad \times \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{n_1} q_p(k-r) \pmod{p}. \end{aligned} \quad (1.13)$$

Proof. Let n be a nonnegative integer with $n \equiv n_0 - pn_1 \pmod{p(p-1)}$. Applying (1.10) with $m = n_0 - p$ we obtain

$$F_p(n, r) \equiv \frac{(-1)^{n^*}}{n^*!} B_{n^*}^{(p-n_0)}(-r) \equiv -(p-1-n^*)! B_{n^*}^{(p-n_0)}(-r) \pmod{p},$$

where $n^* = \{-n\}_{p-1}$.

If $n_0 \leq n_1$, then $n^* = n_1 - n_0$ and hence

$$B_{n_1-n_0}^{(p-n_0)}(-r) \equiv (-1)^{n_1-n_0} (n_1 - n_0)! F_p(n, r) \pmod{p},$$

which implies (1.11) with the help of (1.2).

Now we consider the case $n_0 > n_1$. Clearly $n^* = n_1 - n_0 + p - 1$ and $p - 1 - n^* = n_0 - n_1$. Therefore

$$F_p(n, r) \equiv -(n_0 - n_1)! B_{n_1-n_0+p-1}^{(p-n_0)}(-r) \pmod{p}.$$

The case $n_1 = 0$ of this, together with (1.3), yields (1.12). When $n_1 > 0$, combining the last congruence with (1.4) we obtain (1.13). \square

Corollary 1.3. *Let p be a prime and let $n \in \mathbb{Z}^+$. Then $\text{ord}_p(C_p(n, r)) = \lfloor (n-1)/(p-1) \rfloor$ for at least $p - n^* \geq 2$ values of $r \in \{0, \dots, p-1\}$, where $n^* = \{-n\}_{p-1}$.*

Proof. For any $r \in \mathbb{Z}$, $\text{ord}_p(C_p(n, r)) = \lfloor (n-1)/(p-1) \rfloor$ if and only if $F_p(n, r) \not\equiv 0 \pmod{p}$. By Theorem 1.2,

$$F_p(n, r) \equiv \frac{(-1)^{n^*}}{n^*!} B_{n^*}^{(p-\{n\}_p)}(-r) \pmod{p} \quad \text{for all } r = 0, \dots, p-1.$$

Recall that $B_{n^*}^{(p-\{n\}_p)}(x) \in \mathbb{Z}_p[x]$ is monic and of degree n^* . Also, a polynomial of degree n^* over the field $\mathbb{Z}/p\mathbb{Z}$ cannot have more than n^* distinct zeroes in the field (cf. [IR, p.39]). So the congruence equation $F_p(n, r) \equiv 0 \pmod{p}$ has at most n^* solutions with $r \in \{0, \dots, p-1\}$. This yields the desired result. \square

Corollary 1.4. *Let p be a prime, and let $n \in \mathbb{N}$ and $n^* = \{-n\}_{p-1}$. Then*

$$(-1)^n F_p(n, 0) \equiv \bar{S}(n^* + \{n\}_p, \{n\}_p) \equiv \frac{B_{n^*}^{(m)}}{n^*!} \pmod{p}, \quad (1.14)$$

where m is any nonnegative integer with $m + n \equiv 0 \pmod{p}$. Also,

$$(-1)^n F_p(pn + p - 1, r) \equiv \frac{B_{n^*}(-r)}{n^*!} \equiv -(p-1-n^*)! B_{n^*}(r+1) \pmod{p} \quad (1.15)$$

for all $r \in \mathbb{Z}$, and in particular

$$\binom{2p-1}{p+r} + (-1)^p \binom{2p-1}{r} \equiv (-1)^r p^2 B_{p-2}(-r) \pmod{p^3} \quad (1.16)$$

for every $r = 0, \dots, p-1$.

Proof. Applying Theorem 1.2 with $r = 0$ we immediately get (1.14).

As $pn + p - 1 \equiv -1 \pmod{p}$ and $n^* = \{-(pn + p - 1)\}_{p-1}$, by the second part of Theorem 1.2 and the identity $(-1)^{n^*} B_{n^*}(x) = B_{n^*}(1 - x)$, whenever $r \in \mathbb{Z}$ we have

$$\begin{aligned} (-1)^{n^*} F_p(pn + p - 1, r) &\equiv \frac{B_{n^*}(-r)}{n^*!} \equiv (-1)^{n^*+1} (p-1-n^*)! B_{n^*}(-r) \\ &\equiv - (p-1-n^*)! B_{n^*}(r+1) \pmod{p} \end{aligned}$$

and hence (1.15) holds.

Now let $r \in \{0, \dots, p-1\}$. By (1.15) in the case $n = 1$,

$$-F_p(2p-1, r) \equiv -(p-1-(p-2))! B_{p-2}(r+1) \pmod{p}$$

and hence

$$F_p(2p-1, r) \equiv B_{p-2}(1 - (-r)) = (-1)^{p-2} B_{p-2}(-r) \pmod{p}$$

which is equivalent to (1.16). We are done. \square

Let p be an odd prime, and let h_p and h_p^+ denote the class numbers of the cyclotomic field $\mathbb{Q}(\zeta_p)$ and its maximal real subfield $\mathbb{Q}(\zeta_p + \zeta_p^{-1})$ respectively, where ζ_p is a primitive p -th root of unity in the complex field \mathbb{C} . It is well known that $h_p^- = h_p/h_p^+$ is an integer. If p divides none of the numerators of the Bernoulli numbers $B_0, B_2, \dots, B_{p-3} \in \mathbb{Z}_p$, then p is said to be a *regular* prime. In 1850 E. Kummer proved that

$$\begin{aligned} p \nmid h_p &\iff p \nmid h_p^- \iff p \text{ is regular} \\ &\implies x^p + y^p = z^p \text{ has no integer solution with } xyz \neq 0. \end{aligned}$$

Furthermore,

$$h_p^- \equiv \prod_{0 < n \leq (p-3)/2} \left(-\frac{B_{2n}}{4n} \right) \pmod{p}$$

by the proof of Theorem 5.16 in [Wa, p. 62].

Corollary 1.5. *Let p be a prime.*

(i) *For every $n = 2, \dots, p$ we have*

$$\sum_{k=1}^n (-1)^{pk-1} \binom{pn-1}{pk-1} \equiv (n-1)! B_{p-n} p^n \pmod{p^{n+1}}. \quad (1.17)$$

(ii) *Suppose that $p > 3$. Then p does not divide the class number h_p of the p -th cyclotomic field $\mathbb{Q}(\zeta_p)$, if and only if*

$$\text{ord}_p \left(\sum_{k=1}^n (-1)^k \binom{pn-1}{pk-1} \right) = n \text{ for all } n = 3, 5, \dots, p-2.$$

Also,

$$\begin{aligned} & \sum_{k=1}^{(p-1)/2} (-1)^{k-1} \binom{p(p-1)/2-1}{pk-1} \\ & \equiv \llbracket 4 \mid p+1 \rrbracket (-1)^{(h(-p)+1)/2} h(-p) p^{(p-1)/2} \pmod{p^{(p+1)/2}}, \end{aligned} \quad (1.18)$$

where $h(-p)$ is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$.

Proof. (i) Let $n \in \{2, \dots, p\}$. Then $\lfloor (pn-1-1)/(p-1) \rfloor = n$ and hence

$$F_p(pn-1, -1) = (-p)^{-n} C_p(pn-1, -1) = (-p)^{-n} \sum_{k=1}^n \binom{pn-1}{pk-1} (-1)^{pk-1}.$$

By Corollary 1.4, $(-1)^n F_p(pn-1, -1)$ is congruent to

$$(p-1 - \{-(n-1)\}_{p-1})! B_{\{-(n-1)\}_{p-1}} (-1+1) = (n-1)! B_{p-n}$$

modulo p . Therefore (1.17) holds.

(ii) In view of part (i),

$$\begin{aligned} & \text{ord}_p \left(\sum_{k=1}^n (-1)^k \binom{pn-1}{pk-1} \right) = n \text{ for } n = 3, 5, \dots, p-2 \\ & \iff B_{p-n} \not\equiv 0 \pmod{p} \text{ for } n = 3, 5, \dots, p-2 \\ & \iff p \text{ is regular} \\ & \iff h_p \not\equiv 0 \pmod{p}. \end{aligned}$$

Taking $n = (p-1)/2$ in (1.17) we get

$$\begin{aligned} & \sum_{k=1}^{(p-1)/2} (-1)^{k-1} \binom{p(p-1)/2-1}{pk-1} \\ & \equiv \frac{((p-1)/2)!}{(p-1)/2} p^{(p-1)/2} B_{(p+1)/2} \pmod{p^{(p+1)/2}}. \end{aligned}$$

If $p \equiv 1 \pmod{4}$, then $B_{(p+1)/2} = 0$ since $(p+1)/2 \in \{3, 5, \dots\}$. If $p \equiv 3 \pmod{4}$, then $h(-p) \equiv -2B_{(p+1)/2} \pmod{p}$ (cf. [IR, p. 238]), and $((p-1)/2)! \equiv (-1)^{(h(-p)+1)/2} \pmod{p}$ by Mordell [M]. So (1.18) follows from the above. This concludes the proof. \square

Remark. Let p be an odd prime. If $p \geq 5$, then (1.17) in the case $n = 2$ reduces to Wolstenholme's congruence $\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}$ since $B_{p-2} = 0$. Taking $n = 3$ in (1.17) we get

$$\binom{3p-1}{p-1} - \binom{3p-1}{2p-1} + \binom{3p-1}{3p-1} \equiv 2B_{p-3}p^3 \pmod{p^4};$$

as $\binom{3p-1}{2p-1} = 2\binom{3p-1}{p-1}$ this yields the congruence

$$\binom{3p-1}{p-1} \equiv 1 - 2p^3B_{p-3} \pmod{p^4}.$$

This was first obtained by J.W.L. Glaisher (cf. [G1, p. 21] and [G2, p. 323]) who showed that

$$\binom{pn-1}{p-1} \equiv 1 - \frac{n(n-1)}{3}p^3B_{p-3} \pmod{p^4} \quad \text{for } n = 1, 2, 3, \dots$$

Corollary 1.6. *Let p be an odd prime, and let $n \in \{3, \dots, p\}$ and $r \in \mathbb{Z}$. Then*

$$F_p(pn-2, r) \equiv -n! \left(\frac{B_{p-n+1}(-r)}{n-1} + (r+1) \frac{B_{p-n}(-r)}{n} \right) \pmod{p}. \quad (1.19)$$

Proof. Clearly $\{-(pn-2)\}_{p-1} = p-n+1$. By Theorem 1.2, $F_p(pn-2, r)$ is congruent to

$$-(p-1-(p-n+1))!B_{p-n+1}^{(2)}(-r) = -(n-2)!B_{p-n+1}^{(2)}(-r)$$

modulo p .

Let $m = p - n + 1$. By [PS, (2.14)] or [SP, (1.12)],

$$\begin{aligned} & \frac{(-1)^m}{m} \sum_{k=0}^m \binom{m}{k} B_k B_{m-k}(x) - \frac{B_m(1-x)}{m} B_0 \\ &= - \sum_{k=0}^1 \binom{1}{k} B_{1-k}(x) B_{m-1+k}(1-x) - B_1 B_{m-1}(1-x) \\ &= -B_1(x) B_{m-1}(1-x) - B_0(x) B_m(1-x) - B_1 B_{m-1}(1-x) \\ &= (-1)^m ((B_1(x) + B_1) B_{m-1}(x) - B_m(x)) \\ &= (-1)^m ((x-1) B_{m-1}(x) - B_m(x)). \end{aligned}$$

It follows that

$$\begin{aligned}
B_m^{(2)}(-r) &= \sum_{k=0}^m \binom{m}{k} B_k B_{m-k}(-r) \\
&= (1-m)B_m(-r) + m(-r-1)B_{m-1}(-r) \\
&\equiv (1+n-1)B_{p-n+1}(-r) - (r+1)(-n+1)B_{p-n}(-r) \\
&\equiv n(n-1) \left(\frac{B_{p-n+1}(-r)}{n-1} + (r+1) \frac{B_{p-n}(-r)}{n} \right) \pmod{p}.
\end{aligned}$$

Combining the above we immediately obtain (1.19). \square

By Theorem 1.1 or 1.2, for any prime p the Fleck quotient $F_p(n, r)$ (with $n \in \mathbb{N}$ and $r \in \mathbb{Z}$) modulo p only depends on p and r and the remainder of n modulo $p(p-1)$. This observation can be further extended as follows.

Theorem 1.3. *Let p be a prime, and let $a, l, n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then*

$$\begin{aligned}
&\sum_{k=0}^n \binom{n}{k} (-1)^k F_p(kp^a(p-1) + l, r) \\
&\equiv 0 \pmod{p^{an + \lceil (n-l^*)/(p-1) \rceil}},
\end{aligned} \tag{1.20}$$

where $l^* = \{-l\}_{p-1}$ and $\lceil \cdot \rceil$ is the ceiling function.

The following consequence is somewhat similar to Kummer's congruence for Bernoulli numbers (cf. [IR, pp. 238–241]).

Corollary 1.7. *Let p be a prime, and let $a, l \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then*

$$\begin{aligned}
F_p(p^a(p-1) + l, r) &\equiv F_p(l, r) \pmod{p^a}, \\
F_p(2p^a(p-1) + l, r) &\equiv 2F_p(p^a(p-1) + l, r) - F_p(l, r) \pmod{p^{2a}}, \\
F_p(3p^a(p-1) + l, r) &\equiv 3F_p(2p^a(p-1) + l, r) - 3F_p(p^a(p-1) + l, r) \\
&\quad + F_p(l, r) \pmod{p^{3a}}.
\end{aligned}$$

Proof. Simply apply (1.20) with $n = 1, 2, 3$. \square

Let p be a prime, and let $a \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$. In 1977 C. S. Weisman [We] extended Fleck's result by showing that if $n \geq p^{a-1}$ then

$$C_{p^a}(n, r) \equiv 0 \pmod{p^{\lfloor (n-p^{a-1})/\varphi(p^a) \rfloor}},$$

where φ is Euler's totient function. In view of this, we define the *generalized Fleck quotient*

$$F_{p^a}(n, r) = (-p)^{-\lfloor (n-p^{a-1})/\varphi(p^a) \rfloor} C_{p^a}(n, r) + \llbracket n < p^{a-1} \rrbracket \in \mathbb{Z}.$$

Note that $F_{p^a}(n, r) \equiv 1 \pmod{p}$ for $n = 0, \dots, p^{a-1} - 1$.

Theorem 1.4. *Let p be a prime, and let $a, n \in \mathbb{Z}^+$ with $n \geq p^{a-1}$.*

(i) *For any $r \in \mathbb{Z}$ we have*

$$F_{p^a}(n, r) \equiv \sum_{k=0}^d \binom{r+k-1}{k} F_{p^a}(n+k, 0) \pmod{p}, \quad (1.21)$$

where $d = \{p^{a-1} - 1 - n\}_{\varphi(p^a)}$ is the least nonnegative integer with $n + d \equiv p^{a-1} - 1 \pmod{\varphi(p^a)}$.

(ii) *We have*

$$\text{ord}_p(C_{p^a}(n, r)) = \left\lfloor \frac{n - p^{a-1}}{\varphi(p^a)} \right\rfloor \quad (\text{i.e., } p \nmid F_{p^a}(n, r)) \quad \text{for some } r \in \mathbb{Z}. \quad (1.22)$$

If $n \geq 2p^{a-1}$, then

$$F_{p^a}(n + p^a(p-1), r) \equiv F_{p^a}(n, r) \pmod{p} \quad \text{for all } r \in \mathbb{Z}. \quad (1.23)$$

In view of the first congruence in Corollary 1.7 and the last congruence in Theorem 1.4, we propose the following conjecture.

Conjecture 1.1. *Let p be a prime, and let $a, b, n \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$. If $n \geq 2p^{a+b-2}$, then*

$$F_{p^a}(n + \varphi(p^{a+b}), r) \equiv F_{p^a}(n, r) \pmod{p^b}.$$

Theorems 1.1, 1.2 and 1.3 will be proved in Sections 2, 3 and 4 respectively. In Section 5 we will first give a new proof of Weisman's congruence via roots of unity, and then establish Theorem 1.4.

2. PROOF OF THEOREM 1.1

Lemma 2.1. *Let p be a prime, and let $n \in \mathbb{N}$ and $n^* = \{-n\}_{p-1}$. Define $G(n) = \sum_{a=1}^{p-1} a^n \zeta_p^a$ and $\pi = 1 - \zeta_p$, where ζ_p is a primitive p -th root of unity in the complex field \mathbb{C} . Then*

$$G(n) \equiv (-1)^{n^*-1} \sum_{m=n^*}^{p-2} s(m, n^*) \frac{\pi^m}{m!} \pmod{p}, \quad (2.1)$$

where $s(m, 0), \dots, s(m, m)$ are Stirling numbers of the first kind defined by $(x)_m = \sum_{k=0}^m (-1)^{m-k} s(m, k) x^k$.

Proof. Clearly,

$$\begin{aligned}
G(n) &= \sum_{a=1}^{p-1} a^n (1-\pi)^a = \sum_{a=1}^{p-1} a^n \sum_{m=0}^a \binom{a}{m} (-\pi)^m \\
&= \sum_{m=0}^{p-1} \frac{(-\pi)^m}{m!} \sum_{a=1}^{p-1} a^n (a)_m \\
&= \sum_{m=0}^{p-1} \frac{(-\pi)^m}{m!} \sum_{a=1}^{p-1} a^n \sum_{k=0}^m (-1)^{m-k} s(m, k) a^k \\
&= \sum_{m=0}^{p-1} \frac{(-\pi)^m}{m!} \sum_{k=0}^m (-1)^{m-k} s(m, k) \sum_{a=1}^{p-1} a^{n+k}.
\end{aligned}$$

Since

$$1 + x + \cdots + x^{p-1} = \frac{x^p - 1}{x - 1} = \prod_{a=1}^{p-1} (x - \zeta_p^a),$$

we have

$$\frac{p}{\pi^{p-1}} = \prod_{a=1}^{p-1} \frac{1 - \zeta_p^a}{\pi} = \prod_{a=1}^{p-1} \frac{1 - (1-\pi)^a}{\pi} \equiv \prod_{a=1}^{p-1} a \equiv -1 \pmod{\pi}$$

with the help of Wilson's theorem. Note also that

$$\sum_{a=1}^{p-1} a^{n+k} \equiv -\llbracket p-1 \mid n+k \rrbracket \pmod{p}$$

by elementary number theory (see, e.g., [IR, pp. 235–236]). Therefore

$$\begin{aligned}
G(n) &\equiv \sum_{m=0}^{p-2} \frac{\pi^m}{m!} \sum_{k=0}^m (-1)^k s(m, k) (-\llbracket k = n^* \rrbracket) \\
&\equiv (-1)^{n^*-1} \sum_{m=n^*}^{p-2} s(m, n^*) \frac{\pi^m}{m!} \pmod{p}.
\end{aligned}$$

This concludes the proof. \square

Remark. Let p be an odd prime. For each $a \in \mathbb{Z}$ let $\bar{a} = a + p\mathbb{Z} \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Let ω be the Teichmüller character of the multiplicative group $\mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\}$. For $\bar{a} \in \mathbb{F}_p^*$, $\omega(\bar{a})$ is just the $(p-1)$ -th root of unity in the unique unramified extension of the p -adic field \mathbb{Q}_p with $\omega(\bar{a}) \equiv a \pmod{p}$. (See, e.g., [Wa, p. 51].) If ζ_p is a primitive p -th root of unity in the algebraic closure of \mathbb{Q}_p , then for $n \in \mathbb{N}$ and $\pi = 1 - \zeta_p$ we have

$$\sum_{a=1}^{p-1} a^n \zeta_p^a \equiv \sum_{a=1}^{p-1} \omega^n(\bar{a}) \zeta_p^a \equiv -\frac{(-\pi)^{n^*}}{n^*!} \pmod{\pi^{n^*+1}}$$

with $n^* = \{-n\}_{p-1}$, by Stickelberger's congruence for Gauss' sums (cf. [BEW, pp. 344–345]).

Lemma 2.2. *Let p be a prime, and let ζ_p be a primitive p -th root of unity in \mathbb{C} . Let $n = p^a m + n_0 > 0$ with $a \in \mathbb{Z}^+$ and $m, n_0 \in \mathbb{N}$. Then, for any $r \in \mathbb{Z}$ we have*

$$\begin{aligned} & \pi^{-p^a m} C_p(n, r) - \llbracket p-1 \mid m \rrbracket C_p(n_0, r) \\ & \equiv \frac{G(p^a m)}{p} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{p^a m^*} \pmod{p^{a-1} \pi^{\min\{n_0+1, p-1\}}}, \end{aligned}$$

where $\pi = 1 - \zeta_p$ and $m^* = \{-m\}_{p-1}$.

Proof. Let $j \in \{1, \dots, p-1\}$. Then

$$\left(\frac{1 - \zeta_p^j}{\pi} \right)^m = \left(\frac{1 - (1 - \pi)^j}{\pi} \right)^m = \left(\sum_{i=1}^j \binom{j}{i} (-\pi)^{i-1} \right)^m = j^m + \beta_j \pi,$$

where β_j is a suitable element in the ring $\overline{\mathbb{Z}}$ of algebraic integers. For $i = 0, 1, \dots$, if

$$\left(\frac{1 - \zeta_p^j}{\pi} \right)^{p^i m} = j^{p^i m} + p^i \pi \beta_j^{(i)}$$

for some $\beta_j^{(i)} \in \overline{\mathbb{Z}}$, then

$$\left(\frac{1 - \zeta_p^j}{\pi} \right)^{p^{i+1} m} = \left(j^{p^i m} + p^i \pi \beta_j^{(i)} \right)^p = j^{p^{i+1} m} + p^{i+1} \pi \beta_j^{(i+1)}$$

for some $\beta_j^{(i+1)} \in \overline{\mathbb{Z}}$. So

$$\left(\frac{1 - \zeta_p^j}{\pi} \right)^{p^a m} \equiv j^{p^a m} \pmod{p^a \pi}.$$

Observe that

$$p C_p(n, r) = \sum_{j=0}^{p-1} \zeta_p^{-jr} (1 - \zeta_p^j)^n = \pi^{p^a m} \sum_{j=1}^{p-1} \zeta_p^{-jr} \left(\frac{1 - \zeta_p^j}{\pi} \right)^{p^a m} (1 - \zeta_p^j)^{n_0}.$$

As π^{n_0} divides $(1 - \zeta_p^j)^{n_0}$ in the ring $\overline{\mathbb{Z}}$, by the above $\pi^{-p^a m} p C_p(n, r)$ is congruent to

$$\sum_{j=1}^{p-1} \zeta_p^{-jr} j^{p^a m} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k \zeta_p^{jk} = \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k S_{k-r}$$

modulo $p^a \pi^{n_0+1}$, where

$$S_{k-r} = \sum_{j=1}^{p-1} j^{p^a m} \zeta_p^{j(k-r)}.$$

If $k \not\equiv r \pmod{p}$, then

$$\begin{aligned} S_{k-r} &= (k-r)^{-p^a m} \sum_{j=1}^{p-1} (j(k-r))^{p^a m} \zeta_p^{j(k-r)} \\ &\equiv (k-r)^{p^a m^*} \sum_{t=1}^{p-1} t^{p^a m} \zeta_p^t = (k-r)^{p^a m^*} G(p^a m) \pmod{p^{a+1}}. \end{aligned}$$

(Note that if $j(k-r) \equiv t \pmod{p}$ then $(j(k-r))^{p^a} \equiv t^{p^a} \pmod{p^{a+1}}.$)

Choose a primitive root g modulo p . Since

$$(g^{p^a m} - 1) \sum_{j=1}^{p-1} j^{p^a m} = \sum_{j=1}^{p-1} (gj)^{p^a m} - \sum_{t=1}^{p-1} t^{p^a m} \equiv 0 \pmod{p^{a+1}},$$

if $p-1 \nmid m$ then $g^{p^a m} - 1 \not\equiv 0 \pmod{p}$ and so $\sum_{j=1}^{p-1} j^{p^a m} \equiv 0 \pmod{p^{a+1}}$. Thus, when $k \equiv r \pmod{p}$ we have

$$S_{k-r} = \sum_{j=1}^{p-1} j^{p^a m} \equiv (p-1) \llbracket p-1 \mid m \rrbracket \pmod{p^{a+1}}.$$

Recall that $p/\pi^{p-1} \equiv -1 \pmod{\pi}$. In view of the above,

$$\begin{aligned} &\pi^{-p^a m} p C_p(n, r) - \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{p^a m^*} G(p^a m) \\ &\equiv \sum_{\substack{k=0 \\ p \mid k-r}}^{n_0} \binom{n_0}{k} (-1)^k \left(\llbracket p-1 \mid m \rrbracket (p-1) - (k-r)^{p^a m^*} G(p^a m) \right) \\ &\equiv C_p(n_0, r) \llbracket p-1 \mid m \rrbracket p \pmod{p^a \pi^{\min\{n_0+1, p-1\}}}, \end{aligned}$$

where we have noted that if $p-1 \mid m$ (i.e., $m^* = 0$) then

$$p-1 - G(p^a m) \equiv p - \sum_{t=0}^{p-1} \zeta_p^t = p - \frac{1 - \zeta_p^p}{1 - \zeta_p} = p \pmod{p^{a+1}}.$$

Therefore the desired congruence follows. \square

Proof of Theorem 1.1. In the case $n = 0$, (1.2) holds since $n_1 = n_0 = 0$ and $F_p(n, r) = -pC_p(0, r) + 1$. Below we assume $n > 0$.

Let ζ_p be a primitive p -th root of unity in \mathbb{C} , and set $\pi = 1 - \zeta_p$. By Lemma 2.2 in the case $a = 1$,

$$\begin{aligned} & \pi^{-p\lfloor n/p \rfloor} C_p(n, r) - \llbracket n_1 = 0 \rrbracket C_p(n_0, r) \\ & \equiv \frac{G(p\lfloor n/p \rfloor)}{p} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{pn_1} \pmod{\pi^{\min\{n_0+1, p-1\}}}. \end{aligned}$$

In view of Lemma 2.1,

$$G\left(p \left\lfloor \frac{n}{p} \right\rfloor\right) \equiv G\left(\left\lfloor \frac{n}{p} \right\rfloor\right) \equiv (-1)^{n_1-1} \sum_{m=n_1}^{p-2} s(m, n_1) \frac{\pi^m}{m!} \pmod{p}.$$

If $n_0 > n_1$, then

$$\sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{pn_1} \equiv \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{n_1} = 0 \pmod{p},$$

where we have applied Fermat's little theorem and Euler's identity (mentioned in Section 1). Therefore

$$\begin{aligned} & \pi^{-p\lfloor n/p \rfloor} C_p(n, r) - \llbracket n_1 = 0 \rrbracket C_p(n_0, r) \\ & \equiv \frac{(-1)^{n_1-1}}{p} \sum_{m=n_1}^{p-2} s(m, n_1) \frac{\pi^m}{m!} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{pn_1} \\ & \quad \pmod{\pi^{\llbracket n_0 > n_1 \rrbracket \min\{n_0+1, p-1\}}}. \end{aligned}$$

Recall that $-p/\pi^{p-1} \equiv 1 \pmod{\pi}$. Since $s(n_1, n_1) = 1$ and

$$\frac{p^{\llbracket n_0 \leq n_1 \rrbracket}}{\pi^{n_1}} \pi^{\llbracket n_0 > n_1 \rrbracket \min\{n_0+1, p-1\}} \equiv 0 \pmod{\pi},$$

by the above we have

$$\begin{aligned} & \frac{p^{\llbracket n_0 \leq n_1 \rrbracket} C_p(n, r)}{\pi^{p\lfloor n/p \rfloor + n_1}} - p^{\llbracket n_0 = 0 \rrbracket} \llbracket n_1 = 0 \rrbracket C_p(n_0, r) \\ & \equiv \frac{(-1)^{n_1-1}/n_1!}{p^{\llbracket n_0 > n_1 \rrbracket}} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{pn_1} \pmod{\pi}. \end{aligned}$$

Note that

$$\left\lfloor \frac{n-1}{p-1} \right\rfloor = \left\lfloor \frac{p\lfloor n/p \rfloor + n_0 - 1}{p-1} \right\rfloor = \frac{p\lfloor n/p \rfloor + n_1}{p-1} - \llbracket n_0 \leq n_1 \rrbracket$$

and hence

$$\begin{aligned} \frac{(-p)^{\llbracket n_0 \leq n_1 \rrbracket} C_p(n, r)}{\pi^{p\lfloor n/p \rfloor + n_1}} &= \frac{C_p(n, r)}{(-p)^{\lfloor (n-1)/(p-1) \rfloor}} \left(\frac{-p}{\pi^{p-1}} \right)^{(p\lfloor n/p \rfloor + n_1)/(p-1)} \\ &\equiv F_p(n, r) \pmod{\pi}. \end{aligned}$$

In view of the above,

$$\begin{aligned} &(-1)^{\llbracket n_0 \leq n_1 \rrbracket} F_p(n, r) - \llbracket n_0 > n_1 = 0 \rrbracket C_p(n_0, r) \\ &\equiv \frac{(-1)^{n_1-1}/n_1!}{p^{\llbracket n_0 > n_1 \rrbracket}} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{pn_1} \pmod{\pi}. \end{aligned}$$

As the rational p -adic integer

$$\begin{aligned} D &= F_p(n, r) - \llbracket n_0 > n_1 = 0 \rrbracket C_p(n_0, r) \\ &\quad - \frac{(-1)^{n_1}}{(-p)^{\llbracket n_0 > n_1 \rrbracket} \cdot n_1!} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{pn_1} \end{aligned}$$

is divisible by π , we have $D^{p-1} \equiv 0 \pmod{p}$ and hence $D \equiv 0 \pmod{p}$.

Thus

$$\begin{aligned} &F_p(n, r) - \llbracket n_0 > n_1 = 0 \rrbracket C_p(n_0, r) \\ &\equiv \frac{(-1)^{n_1}}{(-p)^{\llbracket n_0 > n_1 \rrbracket} \cdot n_1!} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{pn_1} \pmod{p}. \end{aligned} \quad (2.2)$$

In the case $n_0 \leq n_1$, (2.2) reduces to (1.2). When $n_0 > n_1 = 0$, (2.2) yields (1.3) since $C_p(n_0, r) = (-1)^{\{r\}_p} \binom{n_0}{\{r\}_p}$ and $\sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k = (1-1)^{n_0} = 0$.

Now assume that $n_0 > n_1 > 0$. As $\sum_{k=0}^{n_0} \binom{n_0}{k} (k-r)^{n_1} = 0$ by Euler's identity, (2.2) implies that

$$F_p(n, r) \equiv \frac{(-1)^{n_1-1}}{n_1!} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k \frac{(k-r)^{pn_1} - (k-r)^{n_1}}{p} \pmod{p}.$$

If $n_1 = 1$, then

$$\frac{(k-r)^{pn_1} - (k-r)^{n_1}}{p} = (k-r)^{n_1} n_1 q_p(k-r);$$

if $n_1 \geq 2$ and $k \equiv r \pmod{p}$, then

$$\frac{(k-r)^{pn_1} - (k-r)^{n_1}}{p} \equiv 0 \equiv (k-r)^{n_1} n_1 q_p(k-r) \pmod{p};$$

if $a = k-r \not\equiv 0 \pmod{p}$, then

$$\frac{(k-r)^{pn_1} - (k-r)^{n_1}}{p} = a^{n_1} \frac{(1+p \cdot q_p(a))^{n_1} - 1}{p} \equiv a^{n_1} n_1 q_p(a) \pmod{p}.$$

Therefore (1.4) follows.

The proof is now complete. \square

3. PROOF OF THEOREM 1.2

The following lemma is a refinement of an induction technique used by Sun [S06].

Lemma 3.1. *Let p be a prime, and let $n \in \mathbb{N}$ with $n \geq p$. Then*

$$F_p(n, r) \equiv - \sum_{j=1}^{p-1} \frac{1}{j} \sum_{i=0}^{j-1} F_p(n-p+1, r-i) \pmod{p}. \quad (3.1)$$

Proof. Set $n' = n - (p-1) > 0$. By the Chu-Vandermonde convolution identity (cf. [GKP, (5.27)]),

$$\begin{aligned} F_p(n, r) &= (-p)^{-\lfloor (n-1)/(p-1) \rfloor} \sum_{\substack{0 \leq k \leq n \\ k \equiv r \pmod{p}}} \sum_{j=0}^k \binom{p-1}{j} \binom{n'}{k-j} (-1)^k \\ &= -\frac{1}{p} \sum_{j=0}^{p-1} \binom{p-1}{j} (-p)^{-\lfloor (n'-1)/(p-1) \rfloor} \sum_{\substack{j \leq k \leq n \\ p \mid k-r}} \binom{n'}{k-j} (-1)^k \\ &= -\frac{1}{p} \sum_{j=0}^{p-1} \binom{p-1}{j} (-1)^j F_p(n', r-j). \end{aligned}$$

For any $j = 0, \dots, p-1$, clearly

$$\begin{aligned} \binom{p-1}{j} (-1)^j &= \prod_{0 < i \leq j} \left(1 - \frac{p}{i}\right) \\ &\equiv 1 - \sum_{0 < i \leq j} \frac{p}{i} \equiv (-1)^{p-1} + p \sum_{j < k < p} \frac{1}{k} \pmod{p^2}. \end{aligned}$$

(Note that $2 \sum_{k=1}^{p-1} 1/k = \sum_{k=1}^{p-1} (1/k + 1/(p-k)) \equiv 0 \pmod{p}$.) Also,

$$\sum_{j=0}^{p-1} F_p(n', r-j) = (-p)^{-\lfloor (n'-1)/(p-1) \rfloor} \sum_{k=0}^{n'} \binom{n'}{k} (-1)^k = 0.$$

Therefore

$$F_p(n, r) \equiv - \sum_{j=0}^{p-1} \sum_{j < k < p} \frac{F_p(n', r-j)}{k} = - \sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=0}^{k-1} F_p(n', r-j) \pmod{p}.$$

This proves (3.1). \square

Proof of Theorem 1.2. (i) Suppose $m \geq 0$. Then

$$\begin{aligned}
& \sum_{k=0}^{n^*} \bar{S}(m+n^*-k, m) \frac{(-r)^k}{k!} \\
&= [x^{m+n^*}] \sum_{l=m}^{\infty} \bar{S}(l, m) x^l \sum_{k=0}^{\infty} \frac{(-rx)^k}{k!} \\
&= [x^{m+n^*}] (e^x - 1)^m e^{-rx} = [x^{n^*}] \left(\frac{e^x - 1}{x} \right)^m e^{-rx} \\
&= [x^{m+n^*}] \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} e^{(k-r)x} = \sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \frac{(k-r)^{m+n^*}}{(m+n^*)!}.
\end{aligned}$$

By the identity (2.4) of Sun [S03], for any $l = 0, 1, \dots$ we have

$$\begin{aligned}
\sum_{k=0}^m \binom{m}{k} (-1)^{m-k} (k+l)^{m+n^*} &= \sum_{j=0}^l \binom{l}{j} (m+j)! S(m+n^*, m+j) \\
&= \sum_{j=0}^{n^*} \binom{l}{j} (m+j)! S(m+n^*, m+j).
\end{aligned}$$

Thus

$$\sum_{k=0}^m \binom{m}{k} (-1)^{m-k} (k+x)^{m+n^*} = \sum_{j=0}^{n^*} \binom{x}{j} (m+j)! S(m+n^*, m+j)$$

and hence

$$\sum_{k=0}^m \binom{m}{k} (-1)^{m-k} \frac{(k-r)^{m+n^*}}{(m+n^*)!} = \sum_{j=0}^{n^*} \binom{-r}{j} \bar{S}(m+n^*, m+j).$$

If $m \leq 0$, then

$$\frac{B_{n^*}^{(-m)}(-r)}{n^*!} = [x^{n^*}] \left(\frac{x}{e^x - 1} \right)^{-m} e^{-rx} = [x^{n^*}] \left(\frac{e^x - 1}{x} \right)^m e^{-rx}.$$

Note also that

$$\frac{1}{n^*!} = \frac{\prod_{j=1}^{p-1-n^*} (p-j)}{(p-1)!} \equiv (-1)^{n^*+1} (p-1-n^*)! \pmod{p}$$

by Wilson's theorem.

In view of the above, whether $m \geq 0$ or $m \leq 0$, we only need to show that

$$(-1)^n F_p(n, r) \equiv [x^{n^*}] \left(\frac{e^x - 1}{x} \right)^m e^{-rx} \pmod{p}.$$

(ii) All those formal power series $f(x) = \sum_{k=0}^{\infty} a_k x^k$ with $a_k \in \mathbb{Q}$ and $a_0, \dots, a_{n^*} \in \mathbb{Z}_p$ form a ring R_{n^*} under the usual addition and multiplication. In particular, this ring contains

$$e^{-rx} = \sum_{k=0}^{\infty} (-r)^k \frac{x^k}{k!}, \quad \frac{e^x - 1}{x} = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} \quad \text{and} \quad \frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

(Recall that $n^* < p-1$ and $B_0, \dots, B_{n^*} \in \mathbb{Z}_p$.) If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$ belong to R_{n^*} , then

$$\begin{aligned} [x^{n^*}] f(x) g(x)^p &= [x^{n^*}] \sum_{j=0}^{n^*} a_j x^j \left(\sum_{k=0}^{n^*} b_k x^k \right)^p \\ &\equiv [x^{n^*}] \sum_{j=0}^{n^*} a_j x^j \sum_{k=0}^{n^*} b_k^p x^{pk} = a_{n^*} b_0^p \equiv [x^{n^*}] f(x) [x^0] g(x) \pmod{p}. \end{aligned}$$

Consequently, for any $a \in \mathbb{Z}$ we have

$$[x^{n^*}] \left(\frac{e^x - 1}{x} \right)^m e^{ax} \equiv [x^{n^*}] \left(\frac{e^x - 1}{x} \right)^n e^{ax} \pmod{p}$$

since $m \equiv n \pmod{p}$. By this and part (i), it suffices to use induction on n to show that

$$(-1)^n F_p(n, r) \equiv [x^{n^*}] \left(\frac{e^x - 1}{x} \right)^n e^{-rx} \pmod{p}. \quad (3.2)$$

(iii) Obviously

$$(-1)^0 F_p(0, r) = -p C_p(0, r) + 1 \equiv 1 = [x^0] \left(\frac{e^x - 1}{x} \right)^0 e^{-rx} \pmod{p}.$$

So (3.2) holds for $n = 0$.

Suppose that $0 < n \leq p-1$. Then $n^* = p-1-n$ and

$$\begin{aligned} [x^{n^*}] \left(\frac{e^x - 1}{x} \right)^n e^{-rx} &= [x^{p-1}] (e^x - 1)^n e^{-rx} \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} [x^{p-1}] e^{(k-r)x} = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} \frac{(k-r)^{p-1}}{(p-1)!} \\ &\equiv (-1)^{n-1} \sum_{k \not\equiv r \pmod{p}} \binom{n}{k} (-1)^k \pmod{p}. \end{aligned}$$

(To get the last congruence we have applied Wilson's theorem and Fermat's little theorem.) Since

$$- \sum_{k \not\equiv r \pmod{p}} \binom{n}{k} (-1)^k = \sum_{k \equiv r \pmod{p}} \binom{n}{k} (-1)^k = F_p(n, r),$$

the desired (3.2) follows.

Now fix $n \geq p$ and assume that (3.2) holds for smaller values of n . Clearly $n' = n - (p - 1) > 0$ and $\{-n'\}_{p-1} = n^*$. In light of Lemma 3.1,

$$F_p(n, r) \equiv - \sum_{j=1}^{p-1} \frac{1}{j} \sum_{k=0}^{j-1} F_p(n', r - k) \pmod{p}.$$

By the induction hypothesis and part (ii),

$$\begin{aligned} (-1)^{n'} F_p(n', r - k) &\equiv [x^{n^*}] \left(\frac{e^x - 1}{x} \right)^{n'} e^{-(r-k)x} \\ &\equiv [x^{n^*}] \left(\frac{e^x - 1}{x} \right)^{n+1} e^{(k-r)x} \pmod{p}. \end{aligned}$$

Thus $(-1)^{n-1} F_p(n, r)$ is congruent to

$$\begin{aligned} &\sum_{j=1}^{p-1} \frac{1}{j} \sum_{k=0}^{j-1} \left([x^{n^*}] \left(\frac{e^x - 1}{x} \right)^{n+1} e^{(k-r)x} \right) \\ &= [x^{n^*}] \left(\frac{e^x - 1}{x} \right)^{n+1} e^{-rx} \sum_{j=1}^{p-1} \left(\frac{1}{j} \cdot \frac{e^{jx} - 1}{e^x - 1} \right) \\ &= [x^{n^*}] \left(\frac{e^x - 1}{x} \right)^n e^{-rx} \sum_{j=1}^{p-1} \frac{e^{jx} - 1}{jx} \end{aligned}$$

modulo p . This yields

$$\begin{aligned} (-1)^n F_p(n, r) &\equiv - [x^{n^*}] \left(\frac{e^x - 1}{x} \right)^n e^{-rx} \sum_{j=1}^{p-1} \sum_{k=1}^{p-1} \frac{(jx)^{k-1}}{k!} \\ &\equiv [x^{n^*}] \left(\frac{e^x - 1}{x} \right)^n e^{-rx} \pmod{p}, \end{aligned}$$

since $n^* < p - 1$ and $\sum_{j=1}^{p-1} j^{k-1} \equiv -\llbracket p - 1 \mid k - 1 \rrbracket \pmod{p}$.

In view of the above, we have completed the proof. \square

4. PROOF OF THEOREM 1.3

Proof of Theorem 1.3. Let ζ_p be a primitive p -th root of unity in \mathbb{C} , and set $\pi = 1 - \zeta_p$. For any $k = 0, \dots, n$, we have

$$\begin{aligned} pC_p(kp^a(p-1) + l, r) &= \sum_{j=0}^{p-1} \zeta_p^{-jr} (1 - \zeta_p^j)^{kp^a(p-1)+l} \\ &= \sum_{j=1}^{p-1} \zeta_p^{-jr} (1 - \zeta_p^j)^{kp^a(p-1)+l} + \llbracket k = l = 0 \rrbracket \end{aligned}$$

and thus

$$\begin{aligned} &F_p(kp^a(p-1) + l, r) \\ &= (-p)^{-\lfloor (kp^a(p-1)+l-1)/(p-1) \rfloor} C_p(kp^a(p-1) + l, r) + \llbracket k = l = 0 \rrbracket \\ &= -(-p)^{-kp^a - \lfloor (l-1)/(p-1) \rfloor - 1} \sum_{j=1}^{p-1} \zeta_p^{-jr} (1 - \zeta_p^j)^{kp^a(p-1)+l}. \end{aligned}$$

Therefore, for $S_n = \sum_{k=0}^n \binom{n}{k} (-1)^k F_p(kp^a(p-1) + l, r)$ we have

$$S_n = - \sum_{j=1}^{p-1} \zeta_p^{-jr} (1 - \zeta_p^j)^l (-p)^{-\lfloor (l-1)/(p-1) \rfloor - 1} c_{n,j}, \quad (4.1)$$

where

$$\begin{aligned} c_{n,j} &= \sum_{k=0}^n \binom{n}{k} (-1)^k (-p)^{-kp^a} (1 - \zeta_p^j)^{kp^a(p-1)} \\ &= \left(1 - (-p)^{-p^a} (1 - \zeta_p^j)^{p^a(p-1)} \right)^n. \end{aligned}$$

Let $j \in \{1, \dots, p-1\}$. Clearly

$$\left(\frac{1 - \zeta_p^j}{\pi} \right)^{p-1} = \left(\frac{1 - (1 - \pi)^j}{\pi} \right)^{p-1} \equiv j^{p-1} \equiv 1 \pmod{\pi}$$

and hence

$$b_j := \frac{(1 - \zeta_p^j)^{p-1}}{-p} = \left(\frac{1 - \zeta_p^j}{\pi} \right)^{p-1} \frac{\pi^{p-1}}{-p} \equiv 1 \pmod{\pi}.$$

(Recall the congruence $p/\pi^{p-1} \equiv -1 \pmod{\pi}$.) It follows that $b_j^{p^a} \equiv 1 \pmod{p^a\pi}$ and

$$c_{n,j} = \left(1 - b_j^{p^a} \right)^n \equiv 0 \pmod{p^{an}\pi^n}. \quad (4.2)$$

Since $(1 - \zeta_p^j)^l \equiv 0 \pmod{\pi^l}$ and $\text{ord}_p(\pi) = 1/(p-1)$, in view of (4.1) and (4.2) we have

$$\text{ord}_p(S_n) \geq \frac{l+n}{p-1} + an - \left\lfloor \frac{l-1}{p-1} \right\rfloor - 1 = an + \frac{l+n}{p-1} - \frac{l+l^*}{p-1} = an + \frac{n-l^*}{p-1}$$

and hence $\text{ord}_p(S_n) \geq an + \lceil (n-l^*)/(p-1) \rceil$. This proves (1.20). \square

5. ON GENERALIZED FLECK QUOTIENTS

Lemma 5.1. *Let $d, q \in \mathbb{Z}^+$, $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Let ζ_{dq} be a primitive dq -th root of unity in \mathbb{C} . Then*

$$C_{dq}(n, r) = \frac{1}{d} \sum_{k=0}^n \binom{n}{k} C_q(k, r) \sum_{j=0}^{d-1} \zeta_{dq}^{j(k-r)} \left(1 - \zeta_{dq}^j\right)^{n-k}. \quad (5.1)$$

Proof. Note that $\zeta = \zeta_{dq}^d$ is a primitive q -th root of unity. Thus

$$\begin{aligned} & q \sum_{k=0}^n \binom{n}{k} C_q(k, r) \sum_{j=0}^{d-1} \zeta_{dq}^{j(k-r)} \left(1 - \zeta_{dq}^j\right)^{n-k} \\ &= \sum_{k=0}^n \binom{n}{k} \sum_{s=0}^{q-1} \zeta^{-sr} (1 - \zeta^s)^k \sum_{j=0}^{d-1} \zeta_{dq}^{j(k-r)} \left(1 - \zeta_{dq}^j\right)^{n-k} \\ &= \sum_{s=0}^{q-1} \sum_{j=0}^{d-1} \zeta_{dq}^{-(ds+j)r} \sum_{k=0}^n \binom{n}{k} \left(\zeta_{dq}^j (1 - \zeta_{dq}^{ds})\right)^k \left(1 - \zeta_{dq}^j\right)^{n-k} \\ &= \sum_{s=0}^{q-1} \sum_{j=0}^{d-1} \zeta_{dq}^{-(ds+j)r} \left(1 - \zeta_{dq}^{ds+j}\right)^n \\ &= \sum_{t=0}^{dq-1} \zeta_{dq}^{-tr} \left(1 - \zeta_{dq}^t\right)^n = dq C_{dq}(n, r). \end{aligned}$$

So we have (5.1). \square

With the help of Lemma 5.1 we can prove the following result via roots of unity.

Theorem 5.1 (Weisman, 1977). *Let p be a prime, and let $a \in \mathbb{Z}^+$, $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then $F_{p^a}(n, r) \in \mathbb{Z}$.*

Proof. We use induction on a .

The case $a = 1$ reduces to Fleck's result. A proof of Fleck's result via roots of unity was given by A. Granville [Gr].

Now let $a \geq 2$ and assume that $F_{p^{a-1}}(n', r') \in \mathbb{Z}$ for all $n' \in \mathbb{N}$ and $r' \in \mathbb{Z}$. If $n < p^a$, then $\lfloor (n - p^{a-1})/\varphi(p^a) \rfloor \leq 0$ and hence $F_{p^a}(n, r) \in \mathbb{Z}$. Below we suppose $n \geq p^a$ and let ζ_{p^a} be a primitive p^a -th root of unity in \mathbb{C} .

By Lemma 5.1,

$$C_{p^a}(n, r) = \frac{1}{p} \sum_{k=0}^n \binom{n}{k} C_{p^{a-1}}(k, r) \sum_{j=0}^{p-1} \zeta_{p^a}^{j(k-r)} \left(1 - \zeta_{p^a}^j\right)^{n-k}. \quad (5.2)$$

Observe that

$$\prod_{\substack{j=1 \\ p \nmid j}}^{p^a-1} (1 - \zeta_{p^a}^j) = \prod_{\substack{\gamma^{p^a}=1 \\ \gamma^{p^{a-1}} \neq 1}} (1 - \gamma) = \lim_{x \rightarrow 1} \frac{x^{p^a} - 1}{x^{p^{a-1}} - 1} = \frac{p^a}{p^{a-1}} = p.$$

If $p \nmid j$, then $(1 - \zeta_{p^a}^j)/(1 - \zeta_{p^a})$ is a unit in the ring $\mathbb{Z}[\zeta_{p^a}]$ and thus

$$\text{ord}_p(1 - \zeta_{p^a}^j) = \text{ord}_p(1 - \zeta_{p^a}) = \frac{1}{\varphi(p^a)}.$$

By this and the induction hypothesis, for any $k = 0, \dots, n$ we have

$$\begin{aligned} & \text{ord}_p \left(C_{p^{a-1}}(k, r) \sum_{j=0}^{p-1} \zeta_{p^a}^{j(k-r)} \left(1 - \zeta_{p^a}^j\right)^{n-k} \right) \\ & \geq \max \left\{ 0, \left\lfloor \frac{k - p^{a-2}}{\varphi(p^{a-1})} \right\rfloor \right\} + \frac{n-k}{\varphi(p^a)} \\ & = \max \left\{ 0, \frac{pk - p^{a-1}}{\varphi(p^a)} - \left\lfloor \frac{k - p^{a-2}}{\varphi(p^{a-1})} \right\rfloor \right\} + \frac{n-k}{\varphi(p^a)} \\ & = \max \left\{ \frac{n-k}{\varphi(p^a)}, \frac{n - p^{a-1}}{\varphi(p^a)} + \frac{k}{p^{a-1}} - \left\lfloor \frac{k - p^{a-2}}{\varphi(p^{a-1})} \right\rfloor \right\} > \frac{n - p^{a-1}}{\varphi(p^a)}. \end{aligned}$$

(Note that if $k \geq p^{a-1}$ then $k/p^{a-1} \geq 1 > \{(k - p^{a-2})/\varphi(p^{a-1})\}$.) Therefore, from (5.2) we get that

$$\text{ord}_p(C_{p^a}(n, r)) > \frac{n - p^{a-1}}{\varphi(p^a)} - 1 \geq \left\lfloor \frac{n - p^{a-1}}{\varphi(p^a)} \right\rfloor - 1.$$

So $F_{p^a}(n, r) = (-p)^{-\lfloor (n - p^{a-1})/\varphi(p^a) \rfloor} C_{p^a}(n, r) \in \mathbb{Z}$ as desired. \square

Proof of Theorem 1.4. (i) Write $n + d = p^{a-1} - 1 + m\varphi(p^a)$ with $m \in \mathbb{N}$. Then, for any $k = 0, \dots, d$ we have

$$\left\lfloor \frac{n + k - p^{a-1}}{\varphi(p^a)} \right\rfloor = \left\lfloor m - \frac{d - k + 1}{\varphi(p^a)} \right\rfloor = m - 1.$$

Below we use induction on d to show the desired congruence (1.21).

In the case $d = 0$ (i.e., $n - p^{a-1} \equiv -1 \pmod{\varphi(p^a)}$), we have $F_{p^a}(n, r) \equiv F_{p^a}(n, 0) \pmod{p}$ because

$$F_{p^a}(n, i) - F_{p^a}(n, i-1) = (-p)^{-m+1} C_{p^a}(n+1, i) = -p F_{p^a}(n+1, i)$$

for all $i \in \mathbb{Z}$. Furthermore, by a result of Weisman [We] (see also [SW, Theorem 1.5]), $F_{p^a}(n, r) \equiv 1 \pmod{p}$ if $d = 0$.

Now let $d > 0$ and assume that the desired result holds for smaller values of d . Clearly, $(n+1) + (d-1) = p^{a-1} - 1 + m\varphi(p^a)$ and

$$\left\lfloor \frac{n+1+k-p^{a-1}}{\varphi(p^a)} \right\rfloor = m-1 \quad \text{for } k = 0, \dots, d-1.$$

If $r \geq 0$ then

$$C_{p^a}(n, r) - C_{p^a}(n, 0) = \sum_{0 < i \leq r} (C_{p^a}(n, i) - C_{p^a}(n, i-1)) = \sum_{0 < i \leq r} C_{p^a}(n+1, i);$$

if $r < 0$ then

$$\begin{aligned} C_{p^a}(n, r) - C_{p^a}(n, 0) &= \sum_{r < i \leq 0} (C_{p^a}(n, i-1) - C_{p^a}(n, i)) \\ &= - \sum_{r < i \leq 0} C_{p^a}(n+1, i). \end{aligned}$$

Therefore

$$F_{p^a}(n, r) - F_{p^a}(n, 0) = \begin{cases} \sum_{0 < i \leq r} F_{p^a}(n+1, i) & \text{if } r \geq 0, \\ -\sum_{r < i \leq 0} F_{p^a}(n+1, i) & \text{if } r < 0. \end{cases}$$

By the induction hypothesis, whenever $i \in \mathbb{Z}$ we have

$$F_{p^a}(n+1, i) \equiv \sum_{k=0}^{d-1} \binom{i+k-1}{k} F_{p^a}(n+1+k, 0) \pmod{p}.$$

For any $k = 0, \dots, d-1$, if $r \geq 0$

$$\sum_{0 < i \leq r} \binom{i+k-1}{k} = \sum_{j=0}^{r+k-1} \binom{j}{k} = \binom{r+k}{k+1}$$

by an identity of S.-C. Chu (cf. [GKP, (5.10)]); if $r < 0$ then

$$\begin{aligned} - \sum_{r < i \leq 0} \binom{i+k-1}{k} &= (-1)^{k+1} \sum_{r < i \leq 0} \binom{-i}{k} = (-1)^{k+1} \sum_{j=0}^{-r-1} \binom{j}{k} \\ &= (-1)^{k+1} \binom{-r}{k+1} = \binom{r+k}{k+1}. \end{aligned}$$

Thus, by the above, $F_{p^a}(n, r)$ is congruent to

$$F_{p^a}(n, 0) + \sum_{k=0}^{d-1} \binom{r+k}{k+1} F_{p^a}(n+1+k, 0) = \sum_{k=0}^d \binom{r+k-1}{k} F_{p^a}(n+k, 0)$$

modulo p . This concludes the induction proof of (1.21). \square

(ii) In the case $a = 1$, the desired results in Theorem 1.4(ii) follow from Corollaries 1.3 and 1.7.

Now we let $a \geq 2$ and $r \in \mathbb{Z}$. Write $n = p^{a-2}(pn_1 + n_0) + s$ and $r = p^{a-2}(pr_1 + r_0) + t$, where $s, t \in \{0, \dots, p^{a-2}-1\}$, $n_0, r_0 \in \{0, \dots, p-1\}$ and $n_1 \in \mathbb{N}$ and $r_1 \in \mathbb{Z}$.

If $p^{a-1} \leq n < p^a$, then

$$F_{p^a}(n, r) = C_{p^a}(n, r) = \binom{n}{\{r\}_{p^a}} (-1)^{\{r\}_{p^a}},$$

and in particular $\text{ord}_p(C_{p^a}(n, 0)) = 0 = \lfloor (n - p^{a-1})/\varphi(p^a) \rfloor$.

Below we assume that $n \geq 2p^{a-1}$ (i.e., $n_1 \geq 2$). By [SD, Theorem 1.7],

$$F_{p^a}(n, r) \equiv (-1)^t \binom{s}{t} F_{p^2}(pn_1 + n_0, pr_1 + r_0) \pmod{p}.$$

If $p \mid n_1$, or $p-1 \nmid n_1-1$, or $n_0 = r_0 = p-1$, then by [SW, Theorem 1.2] in the case $l = 0$, we have

$$F_{p^2}(pn_1 + n_0, pr_1 + r_0) \equiv (-1)^{r_0} \binom{n_0}{r_0} F_p(n_1, r_1) \pmod{p}$$

and hence $F_{p^a}(n, r) \equiv b_{n,r} F_p(n_1, r_1) \pmod{p}$, where

$$\begin{aligned} b_{n,r} &:= (-1)^{\{r\}_{p^{a-1}}} \binom{\{n\}_{p^{a-1}}}{\{r\}_{p^{a-1}}} = (-1)^{p^{a-2}r_0+t} \binom{p^{a-2}n_0+s}{p^{a-2}r_0+t} \\ &\equiv (-1)^t \binom{s}{t} (-1)^{r_0} \binom{n_0}{r_0} \pmod{p} \quad (\text{by Lucas' theorem (cf. [HS])}). \end{aligned}$$

By Corollary 1.3, there is an $r'_1 \in \mathbb{Z}$ such that $F_p(n_1, r'_1) \not\equiv 0 \pmod{p}$. Thus, if $p \mid n_1$ or $p-1 \nmid n_1-1$, then

$$F_{p^a}(n, p^{a-1}r'_1) \equiv F_p(n_1, r'_1) \not\equiv 0 \pmod{p}.$$

If $n_0 = p-1$, then

$$F_{p^a}(n, p^{a-2}(pr'_1 + p-1)) \equiv (-1)^{p-1} \binom{p-1}{p-1} F_p(n_1, r'_1) \not\equiv 0 \pmod{p}.$$

When $p \nmid n_1$, $p-1 \mid n_1-1$ and $n_0 < r_0$, by applying the second part of [SW, Theorem 1.2] in the case $l = 0$, we have

$$F_{p^2}(pn_1 + n_0, pr_1 + r_0) \equiv \llbracket n_1 > 1 \rrbracket \frac{(-1)^{n_0} n_1}{r_0 \binom{r_0-1}{n_0}} = \frac{(-1)^{n_0} n_1}{r_0 \binom{r_0-1}{n_0}} \pmod{p}$$

and hence

$$F_{p^a}(n, r) \equiv (-1)^{n_0+t} \frac{n_1 \binom{s}{t}}{r_0 \binom{r_0-1}{n_0}} \pmod{p}.$$

In particular, if $p \nmid n_1$, $p-1 \mid n_1-1$ and $n_0 < p-1$, then

$$F_{p^a}(n, p^{a-2}(n_0+1)) \equiv \frac{(-1)^{n_0} n_1}{n_0+1} \not\equiv 0 \pmod{p}.$$

In view of the above, we already have (1.22).

To prove the congruence in (1.23), we should also consider the case $p \nmid n_1$, $p-1 \mid n_1-1$ and $n_0 \geq r_0$. By [SW, Lemmas 3.2 and 3.3],

$$\begin{aligned} & p^{-\lfloor (pn_1+n_0-p)/\varphi(p^2) \rfloor} C_{p^2}(pn_1 + n_0, pr_1 + r_0) \\ & - (-1)^{r_0} \binom{n_0}{r_0} p^{-\lfloor (n_1-1)/(p-1) \rfloor} C_p(n_1, r_1) \\ & \equiv (-1)^{n_1-1} p^{-\lfloor (n_1-1-1)/(p-1) \rfloor} C_p(n_1-1, r_1) (-1)^{n_1+r_0} n_1 \binom{n_0}{r_0} \frac{\sigma_{n_0, r_0}(n_1)}{p} \\ & \equiv -(-1)^{r_0} \binom{n_0}{r_0} p^{-(n_1-1)/(p-1)+1} C_p(n_1-1, r_1) n_1 \frac{\sigma_{n_0, r_0}(n_1)}{p} \pmod{p}, \end{aligned}$$

where

$$\sigma_{n_0, r_0}(n_1) = 1 + (-1)^p \frac{\prod_{1 \leq i \leq p, i \neq p-r_0} (p(n_1-1) + r_0 + i)}{\prod_{1 \leq i \leq p, i \neq p-(n_0-r_0)} (n_0 - r_0 + i)} \equiv 0 \pmod{p}.$$

Therefore

$$\begin{aligned} & F_{p^2}(pn_1 + n_0, pr_1 + r_0) - (-1)^{r_0} \binom{n_0}{r_0} F_p(n_1, r_1) \\ & \equiv (-1)^{r_0} \binom{n_0}{r_0} F_p(n_1-1, r_1) n_1 \frac{\sigma_{n_0, r_0}(n_1)}{p} \pmod{p} \end{aligned}$$

and hence

$$F_{p^a}(n, r) \equiv b_{n, r} \left(F_p(n_1, r_1) + F_p(n_1-1, r_1) n_1 \frac{\sigma_{n_0, r_0}(n_1)}{p} \right) \pmod{p},$$

Observe that $n + p^a(p-1) = p^{a-2}(pn'_1 + n_0) + s$ with $n'_1 = n_1 + p(p-1)$. Clearly $F_p(n'_1, r_1) \equiv F_p(n_1, r_1) \pmod{p}$ by Corollary 1.7, and $\sigma_{n_0, r_0}(n'_1) \equiv \sigma_{n_0, r_0}(n_1) \pmod{p^2}$ if $n_0 \geq r_0$. Thus, by the above, $F_{p^a}(n + p^a(p-1), r) \equiv F_{p^a}(n, r) \pmod{p}$. This concludes the proof. \square

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